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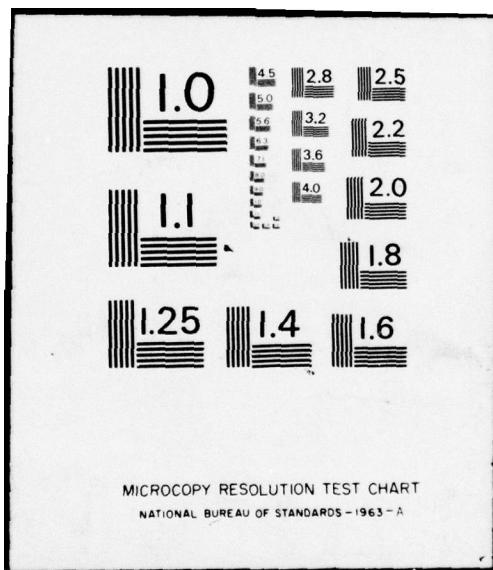
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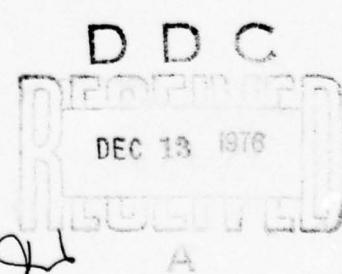
**MACHINE SOLUTIONS  
OF PARTIAL DIFFERENTIAL EQUATIONS IN  
THE NUMERICALLY GENERATED COORDINATE SYSTEMS**

**by**

**Z.U.A. Warsi**

**and**

**J. F. Thompson, Jr.**



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(6) MACHINE SOLUTIONS OF PARTIAL DIFFERENTIAL  
EQUATIONS IN THE NUMERICALLY GENERATED COORDINATE SYSTEMS.

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(10) Z. U. A. Warsi and J. F. Thompson, Jr.

(11) Report Number MSSU-EIRS-ASE-77-1  
ASE-76-153

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(18) AFOSR

(19) TR-76-1241

(15) Under Contracts:  
✓ AF- AFOSR 76-2922-76  
NASA Grant NSG-1242  
and  
NASA-Langley, Grant NGR 25-001-055

(16) P 2304

(17) A3

(11) August 1976

(12) 37P.

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MACHINE SOLUTIONS OF PARTIAL DIFFERENTIAL  
EQUATIONS IN THE NUMERICALLY GENERATED  
COORDINATE SYSTEMS

by

Z. U. A. Warsi and J. F. Thompson, Jr.

**Abstract**

This paper presents a study in depth on the numerically generated body-fitted coordinate systems. All fundamental ideas have been developed from a very basic point of view which leads to a well structured method of coordinate system control in any region of interest. It is also shown how the developed concepts are instrumental in obtaining the machine solutions of the field differential equations in general dependent variables on the generated coordinate systems.

**1. Introduction**

The accuracy of machine solutions of the partial differential equations of mathematical physics depends very strongly on our ability to impose the required boundary conditions as accurately as possible. It is quite natural to envisage a coordinate system in which one coordinate curve passes exactly through the points at which the boundary values of the field equations have been prescribed in the primary Cartesian system ( $x, y$ ). Let the boundary data curve be denoted as  $\eta=\text{constant}$ . To complete a two-dimensional coordinate system we can orient the other coordinate in any desired fashion and call it the curve  $\xi=\text{constant}$ . With this done, the first step is to generate the coordinate curves so as to cover the entire field in which the solution of the partial differential equation is sought. The resulting complexity of the field equations in these coordinates is relatively unimportant.

In fluid mechanics, and also in many other areas of mechanics and physics, the boundary values are usually prescribed on closed curves. For example,

the no-slip condition of viscous flow is prescribed on a closed curve which is the body contour itself. Thus the coordinate  $\eta=\text{constant}$  is chosen as the body contour on which the variations of  $x$  and  $y$  are known in advance. Without loss of generality we can enclose the body by another closed curve on which the field values are either prescribed or can be calculated quite easily. For example in fluid mechanics we can enclose the body, past which the flow takes place, by another curve sufficiently far away from the body, where the free stream values can be prescribed. The idea of numerical coordinate generation is to fill the region between the body and the external boundary curves by coordinate lines in the physical  $(x,y)$  space. Because of the closed region, it is convenient to solve a system of elliptic equations of the simplest type, which are the Laplace equations.

The preceding ideas in one form or another have been used by Winslow [1], Barfield [2], Chu [3], Amsden and Hirt [4], and Godunov and Prokopov [5]. However, the whole concept has been used in a much organized manner to provide a series of solutions in fluid mechanics by Thompson, Thames, and Mastin [6], [7], [8], [9], [10], [11]. References [7] and [11] contain a thorough discussion on the actual computational aspects and computer programming methods both for the coordinate generation and the numerical solutions of the Navier-Stokes equations.

In this paper we summarize the fundamental ideas of the method of coordinate generation which naturally lead to a well structured method of the coordinate system control in the desired regions of the field. Further the use of numerically generated coordinates in the machine solutions of the Navier-Stokes equations written in the contravariant components of the velocity vector is demonstrated.

## 2. Method of Numerical Coordinate Generation

Let  $\xi = \xi(x,y)$  and  $\eta = \eta(x,y)$  be two continuously differentiable functions

of the Cartesian coordinates  $(x, y)$ . The functions

$$\xi(x, y) = C_1, \eta(x, y) = C_2$$

define curvilinear coordinates in the  $xy$ -plane. By varying the constants  $C_1$  and  $C_2$  we can cover the entire  $xy$ -plane by the curvilinear nets or meshes.

The choice of the functions  $\xi$  and  $\eta$  is quite arbitrary because the final outcome of the solution of any field differential equation either in the  $(x, y)$  or  $(\xi, \eta)$  system remains the same. However, the choice of a suitable coordinate system is usually governed by the geometrical configuration of the physical field particularly with a view on satisfying the boundary conditions in as simple and exact manner as possible. Since most of our discussion in this paper is in the context of the viscous flows past finite bodies, the most important boundary condition to be satisfied by the Navier-Stokes equations is the vanishing fluid velocity at a stationary finite body.

For the convenience of satisfying the exact boundary conditions on the body surface, we choose one coordinate line  $\eta = \eta_0 = \text{constant}$  as the body contour. If the body is finite then  $\eta = \eta_0$  is a closed curve. We now enclose the body by another closed curve  $\eta = \eta_\infty$  and call it the outer boundary, where numerically  $\eta_\infty > \eta_0$ . The region  $\eta_0 < \eta < \eta_\infty$  must now be filled by coordinate curves by some method. Because of the closed region under consideration it is natural to specify the determining differential equations for  $\xi$  and  $\eta$  as elliptic equations to be solved under proper boundary conditions at the body and at the external boundary. The simplest elliptic equation is the Laplace equation. With this in mind we pose the problem of solving the Laplace equations for  $\xi$  and  $\eta$  with  $x$  and  $y$  as independent variables under the Dirichlet boundary conditions. Let  $\Gamma_1$  be the curve defining the body contour  $\eta = \eta_0$  and  $\Gamma_2$  be the curve defining the outer boundary  $\eta = \eta_\infty$  in the  $xy$ -plane, Fig. 1. The elliptic

boundary value problem is then

$$\nabla^2 \xi = 0 , \quad (1)$$

$$\nabla^2 \eta = 0 , \quad (2)$$

$$\xi = f_o(x, y) , \eta = \eta_o \text{ on } \Gamma_1 , \quad (3)$$

$$\xi = f_\infty(x, y) , \eta = \eta_\infty \text{ on } \Gamma_2 , \quad (4)$$

To fix ideas we consider the example of plane polar coordinates  $(r, \theta)$ , where it is known that

$$x = r \cos\theta , \quad y = r \sin\theta . \quad (5)$$

It is easy to show that  $\nabla^2 \theta = 0$  but  $\nabla^2 r \neq 0$ . However, the variables defined as

$$\xi = \theta , \quad \eta = \ln r , \quad (6)$$

where

$$x = r \cos\theta = e^{\ln r} \cos\theta = e^\eta \cos\xi , \quad (7)$$

and

$$y = r \sin\theta = e^{\ln r} \sin\theta = e^\eta \sin\xi \quad (8)$$

are such that  $\nabla^2 \xi = 0$  and  $\nabla^2 \eta = 0$ . If we take the inner and outer bounding circles of radii  $r_1$  and  $r_2$  respectively, then

$$\eta_o = \ln r_1 , \eta_\infty = \ln r_2 . \quad (9)$$

In this case we also have

$$f_o = f_\infty = \tan^{-1} \frac{y}{x} .$$

Thus the solution of the elliptic problem (1)-(4) for the circular annulus is

$$\xi = \tan^{-1} \frac{y}{x} , \eta = \frac{1}{2} \ln(x^2 + y^2)$$

The above example illustrates that it should be quite convenient to solve Eqs. (1)-(4) numerically in those cases in which it is possible to specify the values of  $\eta_o$  and  $\eta_\infty$  by simple analytic means. Unfortunately the list of such body and outer boundary shapes and configurations is painfully

short. Thus the method of numerical coordinate generation for arbitrary shaped bodies and outer boundaries through Eqs. (1)-(4) will not be very practical. The source of difficulty lies not so much in the specifications of  $f_0$  and  $f_\infty$  but in the specifications of  $\eta_0$  and  $\eta_\infty$ .

To overcome these difficulties, and also for other important purposes to be described later, we interchange the roles of the dependent and independent variables in Eqs. (1)-(4). Thus using the relations

$$x_\xi = J\eta_y, \quad x_\eta = -J\xi_y, \quad y_\xi = -J\eta_x, \quad y_\eta = J\xi_x, \quad (10)$$

where

$$J = \sqrt{g} = x_\xi y_\eta - x_\eta y_\xi, \quad (11)$$

and the implicit differentiation formulae

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} \quad \text{etc.}, \quad (12)$$

we obtain

$$g_{22} x_{\xi\xi} - 2g_{12} x_{\xi\eta} + g_{11} x_{\eta\eta} = 0, \quad (13)$$

$$g_{22} y_{\xi\xi} - 2g_{12} y_{\xi\eta} + g_{11} y_{\eta\eta} = 0. \quad (14)$$

(Equations (13) and (14) can also be written down directly by first writing  $\phi=\xi$  and then  $\phi=\eta$  in Eq. (A1-33) and then solving them simultaneously). For the definitions of the quantities  $g_{ij}$  refer to Appendix A-1.

The transformation (10)-(12) implies that the body and the external boundary contours in the  $xy$ -plane have been mapped on to the  $\xi\eta$ -plane. In other words we can say that the contours in the  $xy$ -plane have been opened up to form the straight lines  $\eta=\eta_0=\text{const.}$ , and  $\eta=\eta_\infty=\text{const.}$  in the  $\xi\eta$ -plane. This can be achieved by imagining a cut connecting the body and the external boundary in the  $xy$ -plane as shown in Fig. 1, such that all functions and their derivatives are continuous in crossing the cut.

The boundary conditions for Eqs. (13) and (14) are

$$x = F_1(\xi, \eta_o), y = F_2(\xi, \eta_o) \text{ on } \Gamma_1^*, \quad (15)$$

$$x = G_1(\xi, \eta_\infty), y = G_2(\xi, \eta_\infty) \text{ on } \Gamma_2^*, \quad (16)$$

where as shown in Fig. 2,  $\Gamma_1^*$  and  $\Gamma_2^*$  are the images of the body and the external boundary contours in the  $\xi\eta$ -plane. Obviously no boundary conditions can be specified on  $\Gamma_3^*$  and  $\Gamma_4^*$ .

Equations (13) and (14) are quasilinear uniformly elliptic partial differential equations, as proved in Eq. (A2-6) of Appendix-A2.

The appearance of  $\eta_o$  and  $\eta_\infty$  in (15 and (16) is now purely symbolic, denoting the names of the body and the external boundary respectively. Given the body and the external boundary contours, we can always establish the values of  $x$  and  $y$  either graphically or analytically for any desired distribution of the  $\xi$ -values. The  $\eta$ -values can also be chosen arbitrarily to form rectangular meshes in the  $\xi\eta$ -plane.

A question which naturally arises at this stage is this: how to choose the variations of  $\xi$  and  $\eta$  or how to label them in covering the whole  $\xi\eta$ -plane? An answer to this question is that the whole operation can be more practically oriented if we can establish a correspondence between the actual curvilinear coordinate values  $(\xi, \eta)$  and the set of integers. To effect this transition, we introduce new variables  $(\chi, \sigma)$  through a linear transformation

$$\xi = ax + \xi_o, \eta = b\sigma + \eta_o, \quad (17)$$

where  $a$  and  $b$  are constants. Carrying out the transformation (17) in (13) and (14), we have

$$\bar{g}_{22} x_{xx} - 2\bar{g}_{12} x_{x\sigma} + \bar{g}_{11} x_{\sigma\sigma} = 0, \quad (18)$$

$$\bar{g}_{22} y_{xx} - 2\bar{g}_{12} y_{x\sigma} + \bar{g}_{11} y_{\sigma\sigma} = 0, \quad (19)$$

where  $\bar{g}_{ij}$  are functions of  $(\chi, \sigma)$ , viz.,

$$\bar{g}_{11} = x_x^2 + y_x^2, \bar{g}_{12} = x_x x_\sigma + y_x y_\sigma, \bar{g}_{22} = x_\sigma^2 + y_\sigma^2.$$

For these and other definitions refer to Appendix-A1. Here it must be noted that for any function  $F(\xi, \eta)$ ,

$$\begin{aligned} F_x &= a(F_\xi)_{\xi=a\xi_0}, & F_\sigma &= b(F_\eta)_{\eta=b\sigma+\eta_0} \\ F_{xx} &= a^2(F_{\xi\xi})_{\xi=a\xi_0}, & F_{\sigma\sigma} &= b^2(F_{\eta\eta})_{\eta=b\sigma+\eta_0}, \end{aligned} \quad (20a)$$

$$F_{x\sigma} = ab[F_{\xi\eta}]_{\substack{\xi=a\xi_0 \\ \eta=b\sigma+\eta_0}}, \quad (20b)$$

Equations (18) and (19) are exactly of the same form as Eqs. (13) and (14). However a great simplification can be achieved if  $x$  and  $\sigma$  are treated as integers, for then irrespective of the numerical values of  $a$  and  $b$  we can proceed either in the  $\xi$  or  $\eta$  directions through consecutive integers. Further, according to the right hand sides of Eqs. (20 a,b) the first derivatives appearing in Eqs. (18) and (19) can be approximated by forward, backward, or central differences without specifying  $a$  or  $b$ . Similarly, one can approximate the second derivatives in Eqs. (18) and (19) by a central difference. All higher derivatives can similarly be approximated by the proper differences. Under no circumstance should the quantities like  $F_x$  etc. be interpreted as derivatives with respect to integers!

The machine solution of Eqs. (18) and (19) can now be easily performed by employing the proper differences for the derivatives without specifying the step sizes. The same argument is of course true for Eqs. (13) and (14) because they are also homogeneous. However, it must be emphasized that the approach taken here is fundamental in building the essential concepts and establishing the versatility of the method. The reason for developing the

concepts this way, rather than the one originally proposed<sup>†</sup>, is that the method of numerical coordinate generation and the solutions of other differential equations (field equations), either homogeneous or inhomogeneous over the generated meshes, must be considered together and not separately. It will be shown later that the same concepts are applicable when the step sizes in the  $\xi\eta$ -plane are variable and not fixed like  $a$  and  $b$ .

The main utility of numerically generated body-fitted coordinates actually lies in the availability of meshes or nets in the  $\xi\eta$ -plane on which any differential or integral equation representing a field can be solved. First the differential or integral equation must be transformed to a general coordinate system  $(\xi, \eta)$  and then to  $(x, \sigma)$ . If the equation in the  $(x, \sigma)$  variables does not show any explicit dependence on  $a$  and  $b$  then we can make a finite difference approximation of the equations without specifying the step sizes. This statement is equally true for the variable mesh sizes. Later in the paper we have demonstrated that in the case of the Navier-Stokes equations the dependent variables can always be selected in such a way that the step sizes do not appear explicitly. The most important point to mention is that in the midst of these transformations and adjustments of terms, the physical outcome of the solution is not disturbed.

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<sup>†</sup>In their original proposal the authors [6], [7], treated the actual curvilinear coordinates  $(\xi, \eta)$  as integers. The main reason for not introducing the above concept was that the Navier-Stokes equations which they solved had the Cartesian components of velocity as independent variables so that the step sizes  $a$  and  $b$  did not appear in their equations.

### 2.1 Variable Meshes and Coordinate Contraction:

To see the effect of the step size more clearly and to have a method of changing the step sizes, we make a general transformation in the basic elliptic system Eqs. (13) and (14) of the form

$$\xi = f_1(x) + \xi_0, \quad n = f_2(\sigma) + n_0, \quad (21)$$

where

$$\begin{aligned} \xi &= \xi_0 \text{ at } x = 0, \\ \text{and} \quad n &= n_0 \text{ at } \sigma = 0. \end{aligned} \quad (22)$$

Writing

$$\lambda(x) = \frac{df_1}{dx}, \quad \theta(\sigma) = \frac{df_2}{d\sigma}, \quad (23a)$$

$$\frac{dx}{d\xi} = \frac{1}{\lambda}, \quad \frac{d\sigma}{dn} = \frac{1}{\theta}, \quad (23b)$$

so that the derivatives transform as

$$\begin{aligned} x_\xi &= x_x / \lambda, \\ x_n &= x_\sigma / \theta, \\ y_\xi &= y_x / \lambda, \\ y_n &= y_\sigma / \theta, \\ g_{11} &= \bar{g}_{11} / \lambda^2; \quad \bar{g}_{11} = x_x^2 + y_x^2, \\ g_{12} &= \bar{g}_{12} / \theta \lambda; \quad \bar{g}_{12} = x_x x_\sigma + y_x y_\sigma, \\ g_{22} &= \bar{g}_{22} / \theta^2; \quad \bar{g}_{22} = x_\sigma^2 + y_\sigma^2, \\ g &= \bar{g} / \theta^2 \lambda^2; \quad \bar{g} = \bar{g}_{11} \bar{g}_{22} - (\bar{g}_{12})^2, \\ J &= \bar{J} / \theta \lambda; \quad \bar{J} = x_x y_\sigma - x_\sigma y_x. \end{aligned} \quad (24)$$

Further noting that  $\lambda = \lambda(x)$  and  $\theta = \theta(\sigma)$ , we have

$$\begin{aligned} x_{\xi\xi} &= (x_{xx} - \frac{x_x x_x}{\lambda}) / \lambda^2, \\ x_{\xi n} &= x_{x\sigma} / \theta \lambda, \\ x_{nn} &= (x_{\sigma\sigma} - \frac{\sigma_x \sigma_x}{\theta}) / \theta^2, \end{aligned} \quad (25)$$

and similar expressions for the second derivatives of  $y$ .

Substituting (24) and (25) in (13) and (14), we have

$$\bar{g}_{22}x_{xx} - 2\bar{g}_{12}x_{x\sigma} + \bar{g}_{11}x_{\sigma\sigma} = px_x + qx_\sigma , \quad (26)$$

$$\bar{g}_{22}y_{xx} - 2\bar{g}_{12}y_{x\sigma} + \bar{g}_{11}y_{\sigma\sigma} = py_x + qy_\sigma , \quad (27)$$

where

$$p = \frac{\bar{g}_{22}}{\lambda} \lambda_x , \quad q = \frac{\bar{g}_{11}}{\theta} \theta_\sigma . \quad (28)$$

It is a matter of direct verification that the differential equations

$$\nabla^2 x = -R(x, \sigma) , \quad (29)$$

and

$$\nabla^2 \sigma = -S(x, \sigma) , \quad (30)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} , \quad R = (x_x^2 + x_y^2) \lambda_x / \lambda , \quad S = (\sigma_x^2 + \sigma_y^2) \theta_\sigma / \theta$$

transform to the same forms as Eqs. (26) and (27). Differential equations of the form (29) and (30) yield solutions which are called superharmonic if  $R > 0$  and  $S > 0$ . As is proved in Appendix-A3, the superharmonic functions like harmonic functions always satisfy a minimum principle and are bounded from below by the values of the harmonic functions satisfying the same boundary conditions.

As in the case of uniform step sizes, the finite difference approximation of Eqs. (26) or (27) also does not depend on the step sizes. Further, because of the variables  $\lambda$  and  $\theta$  we can now control the mesh spacings both in the  $\xi$  and  $\eta$  directions merely by specifying the functions  $f_1(\chi)$  and  $f_2(\sigma)$ . The functions  $f_1$  and  $f_2$  must be specified as continuous and preferably analytic functions of their arguments.<sup>†</sup> Care should be taken in not interpreting

<sup>†</sup> It is preferable to have both  $f_1(\chi)$  and  $f_2(\sigma)$  analytic in the range  $-\infty < \chi < \infty$  and  $-\infty < \sigma < \infty$  to admit all integral values for  $\chi$  and  $\sigma$ .

$x$  and  $\sigma$  as integers till all their needed derivatives have been obtained in the usual way. Once  $\lambda_x$  and  $\theta_\sigma$  have been obtained and inserted in equations (28), the variables  $x$  and  $\sigma$  can then be interpreted as integers in the solutions of Eqs. (26) and (27). The following example illustrates this point.

Suppose we need to compact the mesh in the neighborhood of the body surface to have a better resolution of the viscous effects in a fluid flow problem. Since no contraction is needed along the  $\xi$ -direction, we set

$$f_1(x) = ax, \quad \xi_0 = 0, \quad (31)$$

where  $a$  = constant.

For the function  $f_2(\sigma)$  let us prescribe an analytic function

$$f_2(\sigma) = b\kappa^{\sigma-1}, \quad (32)$$

where  $\kappa$  and  $b$  are constants. With (31) and (32) we have

$$\lambda = a = \text{const.},$$

$$\theta(\sigma) = b\kappa^{\sigma-1}(1+\sigma\ln\kappa), \quad (33)$$

$$\theta_\sigma = b\kappa^{\sigma-1}(2+\sigma\ln\kappa)\ln\kappa. \quad (34)$$

Therefore from (28),

$$P = 0,$$

$$Q = \frac{\bar{g}_{11}[2\ln\kappa + \sigma(\ln\kappa)^2]}{1+\sigma\ln\kappa}. \quad (35)$$

It is interesting to note that the constant  $b$  does not appear in  $Q$ . Inserting (35) in Eqs. (26) and (27) we get the equations in which, by approximating the derivatives by differences, we can treat both  $x$  and  $\sigma$  as integers. Equations (26) and (27) along with the values of  $P$  and  $Q$  given in (35) have been solved for the case of an ellipse with  $\kappa=1.1$  and the generated coordinates have been shown in Fig. 3. The generated values of  $x$  and  $y$  for this case match quite closely with the available exact conformal solution.

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<sup>†</sup>Note that we could match the generated solution with available exact solution because given the semi-major and minor axes of both contours we could find the constant  $b$ .

### 3. Application to the Navier-Stokes Equations

In this section we demonstrate that in the case of the Navier-Stokes equations the form of the equations whether written in the actual curvilinear coordinates or in the generated curvilinear coordinates remains the same. This result implies that in the solution of the Navier-Stokes equations the step sizes, whether constant or variable, are not to be specified from outside. The effect of the step size is however built into the generated coordinate meshes which have already been obtained a'priori by solving Eqs. (26) and (27).

The Navier-Stokes equations in non-dimensional form written in general curvilinear coordinates  $\xi^i$  and with the contravariant components of the velocity vector ( $v^i$ ) and  $1/\epsilon$  as the Reynolds number are

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} (\sqrt{g} \rho v^i) = 0 , \quad (36)$$

$$\begin{aligned} \frac{\partial}{\partial t} (\rho v^i) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\rho \sqrt{g} v^i v^j) + \Gamma_{jk}^i \rho v^j v^k \\ = - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^j} (p \sqrt{g} g^{ij}) - \Gamma_{jk}^i p g^{jk} \\ + \epsilon \left[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^j} (\sqrt{g} s g^{ij}) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^j} (\sqrt{g} d^{ij}) \right. \\ \left. + s \Gamma_{jk}^i g^{jk} + \Gamma_{jk}^i d^{jk} \right] , \end{aligned} \quad (37)$$

where  $\rho$  is the density (absolute scalar),

$$\begin{aligned} s &= \mu' \text{div } v , \\ d^{ij} &= \mu (g^{ik} v_k^j + g^{jl} v_l^i) , \end{aligned}$$

$\mu$  and  $\mu'$  being the first and second coefficients of viscosity, respectively.

For all other notation and definitions refer to Appendix-A1.

In two dimensions, Eq. (37) represents two equations ( $i=1,2$ ). Introducing the transformation (21) and using (24) it can be shown that

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{1}{\lambda} \bar{\Gamma}_{11}^1 - \frac{1}{\lambda^2} \lambda_x , \\
\Gamma_{22}^2 &= \frac{1}{\theta} \bar{\Gamma}_{22}^2 - \frac{1}{\theta^2} \theta_\sigma , \\
\Gamma_{22}^1 &= \frac{\lambda}{\theta} \bar{\Gamma}_{22}^1 , \\
\Gamma_{11}^2 &= \frac{\theta}{\lambda^2} \bar{\Gamma}_{11}^2 , \\
\Gamma_{12}^1 &= \frac{1}{\theta} \bar{\Gamma}_{12}^1 , \\
\Gamma_{12}^2 &= \frac{1}{\lambda} \bar{\Gamma}_{12}^2 , \\
\Gamma_{11}^1 + \Gamma_{12}^2 &= \frac{1}{2\lambda \bar{g}} \bar{g}_x - \frac{1}{\lambda^2} \lambda_x , \\
\Gamma_{12}^1 + \Gamma_{22}^2 &= \frac{1}{2\theta \bar{g}} \bar{g}_\sigma - \frac{1}{\theta^2} \theta_\sigma , \tag{38}
\end{aligned}$$

where  $\bar{\Gamma}_{jk}^i$  has the same form in  $(x, \sigma)$  as  $\Gamma_{jk}^i$  has in  $(\xi, \eta)$ .

If we now define new contravariant components

$$u^1 = \frac{v^1}{\lambda}, \quad u^2 = \frac{v^2}{\theta} \tag{39}$$

then by substituting (38) and (39) in (36) and (37) the equations retain the original form except that the independent variables are  $(x, \sigma)$  and the  $g$ 's and  $\Gamma$ 's are over barred. At no place  $\lambda$  and  $\theta$  appear explicitly in the equations. Therefore if we treat  $x$  and  $\sigma$  as integers then all the derivatives can be approximated by their corresponding differences.

It was mentioned earlier that by using all these transformations the physical outcome of the solution is not disturbed. The following example is sufficient to demonstrate this fact.

If  $v^1$  and  $v^2$  are the contravariant components of the velocity vector  $\underline{v}$  then the magnitude of the velocity is given by

$$|\underline{v}|^2 = \bar{g}_{11}(v^1)^2 + 2\bar{g}_{12} v^1 v^2 + \bar{g}_{22}(v^2)^2 . \tag{40}$$

Using (24) and (39), the same magnitude is now given by

$$|\underline{v}|^2 = \bar{g}_{11}(u^1)^2 + 2\bar{g}_{12} u^1 u^2 + \bar{g}_{22}(u^2)^2 . \tag{41}$$

It is a matter of direct verification that the Navier-Stokes equations written in any other form, such as in the covariant or cartesian components of the velocity vector, can be solved in the generated coordinate system without any specification of the step sizes. In passing we may note that in two dimensions the relations between the Cartesian and the contravariant components of the velocity vector  $v$  are simply

$$\begin{aligned} u &= x_\xi v^1 + x_n v^2 = x_x u^1 + x_\sigma u^2 , \\ v &= y_\xi v^1 + y_n v^2 = y_x u^1 + y_\sigma u^2 . \end{aligned} \quad (42)$$

#### 4. Truncation Errors

In any finite difference approximation it is imperative to have an estimate of the truncation errors to establish the accuracy and consistency of the difference scheme. To investigate the problem of truncation errors in the present case we consider a function  $F(\xi, n)$  of the coordinates  $(\xi, n)$ . For simplicity consider that  $\xi$  is fixed and  $n$  assumes consecutive values  $n_1, n_2, n_3$ . Let us use the following notation

$$F_i = F(\xi, n_i) , \quad (\frac{\partial F}{\partial n})_{n=n_i} = F_i' , \text{ etc.}, \quad i = 1, 2, 3, .$$

Using Taylor's expansion, we have

$$F_1 = F_2 - (n_2 - n_1)F_2' + \frac{1}{2}(n_2 - n_1)^2 F_2'' - \frac{1}{6}(n_2 - n_1)^3 F_2''' + \dots ,$$

$$F_3 = F_2 + (n_3 - n_2)F_2' + \frac{1}{2}(n_3 - n_2)^2 F_2'' + \frac{1}{6}(n_3 - n_2)^3 F_2''' + \dots .$$

Thus

$$\frac{F_3 - F_1}{n_3 - n_1} - (\frac{\partial F}{\partial n})_2 = \frac{1}{2}(n_1 - 2n_2 + n_3)F_2'' + \frac{[(n_3 - n_2)^3 + (n_2 - n_1)^3]}{6(n_3 - n_1)} F_2''' + \dots . \quad (43)$$

Equation (43) shows the difference between the computed and true derivative, viz., the truncation or the discretization error in the first derivative, in terms of the consecutive values  $n_1, n_2$  and  $n_3$ . To find the order of magnitude of the right hand side in Eq. (43), we introduce the transformation

$$n = f_2(\sigma) + n_0 ,$$

so that

$$\eta_i = f_2(\sigma_i) + \eta_0, \quad i = 1, 2, 3. \quad (44)$$

Using Taylor's expansion of (44) about  $\sigma_2$ , we get

$$\begin{aligned}\eta_1 &= \eta_0 + f_2(\sigma_2) - (\sigma_2 - \sigma_1)(\theta)_2 + \frac{1}{2}(\sigma_2 - \sigma_1)^2(\theta_{\sigma})_2 \\ &\quad - \frac{1}{6}(\sigma_2 - \sigma_1)^3(\theta_{\sigma\sigma})_2 + \dots, \\ \eta_3 &= \eta_0 + f_2(\sigma_2) + (\sigma_3 - \sigma_2)(\theta)_2 + \frac{1}{2}(\sigma_3 - \sigma_2)^2(\theta_{\sigma})_2 \\ &\quad + \frac{1}{6}(\sigma_3 - \sigma_2)^3(\theta_{\sigma\sigma})_2 + \dots,\end{aligned}$$

where  $\theta = \frac{df}{d\sigma}$ , and a subscript 2 on parentheses denotes the value at  $\sigma = \sigma_2$ . It must be emphasized again that  $\theta(\sigma)$  and all its higher derivatives are obtained by differentiation of the continuous function  $f_2(\sigma)$ .

If we now interpret  $\sigma$  as assuming consecutive integral values then

$$\eta_1 - 2\eta_2 + \eta_3 \approx (\theta_{\sigma})_2, \quad (45)$$

and

$$\frac{(\eta_3 - \eta_2)^3 + (\eta_2 - \eta_1)^3}{6(\eta_3 - \eta_1)} \approx \frac{1}{6} [(\theta)_2]^2 + \frac{1}{8} [(\theta_{\sigma})_2]^2. \quad (46)$$

To have a numerical estimate of the terms on the right hand sides of (45) and (46) we have to prescribe an  $f_2(\sigma)$  to determine  $\theta$  and its derivatives. For example the function  $f_2(\sigma)$  as prescribed in (32) shows that both  $\theta$  and  $\theta_{\sigma}$  are of the order of  $b$ , which is implicitly assumed to be a very small number.

The above analysis also points out that the choice of the functions  $f_1$  and  $f_2$  must be made with the following two criteria:

1. Both  $f_1$  and  $f_2$  must be analytic.
2. The values of  $\theta$  and all its derivatives be small for all values of  $x$  and  $\sigma$ .

## 5. Non-Steady Coordinates

Body-fitted coordinates are subject to change with time if the body and/or the external boundary contours change with time. The physical situa-

tions in which the non-steady nature of the coordinates must be considered are, for example, when coordinates are attached to a pulsating body, to a propagating blast or shock wave, free surface, etc.

It will be shown in this section that by using implicit differentiation rules the non-steady nature of the coordinate system can be brought into the time derivative terms of the differential equations which are to be solved on the generated coordinate system. For each time step the coordinates have to be generated or upgraded by using the same equations, viz. (26) and (27) under the changed boundary data. The greatest facility with this approach is that no interpolations are needed between two successive time intervals and the numerical solution proceeds with the same algorithm as when the coordinates are not changing with time.

Referring to Fig. 1, let the point  $P$  away from the body has Cartesian coordinates  $(x, y)$ . In fluid flow problems  $(x, y)$  are the Eulerian variables and represent a fixed point in the physical space. Thus, no matter how the body moves, or the oncoming flow takes place, the total time derivative of  $x$  and  $y$  are zero. That is

$$\frac{dx}{dt} = 0 , \quad \frac{dy}{dt} = 0 . \quad (47)$$

Let  $f(x, y, t)$  be a flow quantity. Then in the body-fitted coordinates if the body and the external boundary are kept fixed, then no matter how the flow takes place, the partial derivative of  $f$  is given by

$$(\frac{\partial f}{\partial t})_{x,y} = (\frac{\partial f}{\partial t})_{\xi,\eta} \quad (48)$$

since in principle  $x = x(\xi, \eta)$ , and  $y = y(\xi, \eta)$ . On the other hand if the body-fitted coordinates are time dependent, then

$$\xi = \xi(x, y, t), \quad \eta = \eta(x, y, t) , \quad (49)$$

from which we deduce that

$$(\frac{\partial f}{\partial t})_{x,y} = (\frac{\partial f}{\partial t})_{\xi,\eta} + f_\xi \xi_t + f_\eta \eta_t , \quad (50)$$

where as before the variable subscripts denote the partial derivatives.

Now from (49), we also have the inversion

$$x = x(\xi, n, t) , \quad y = y(\xi, n, t) . \quad (51)$$

Thus using (47), we have

$$x_t + x_\xi \xi_t + x_n n_t = 0 , \quad (52a)$$

$$y_t + y_\xi \xi_t + y_n n_t = 0 , \quad (52b)$$

which on solution give

$$\xi_t = (y_t x_n - x_t y_n)/J , \quad (53a)$$

$$n_t = (x_t y_\xi - y_t x_\xi)/J . \quad (53b)$$

Substituting (53) in (50), we have

$$\begin{aligned} (\frac{\partial f}{\partial t})_{x,y} &= (\frac{\partial f}{\partial t})_{\xi,n} + \frac{1}{J}(f_n y_\xi - f_\xi y_n)x_t \\ &\quad + \frac{1}{J}(f_\xi x_n - f_n x_\xi)y_t . \end{aligned} \quad (54)$$

Equation (54) can now be used to replace the partial time derivatives in the Navier-Stokes equations written in the Cartesian velocity components ( $U, V$ ), or in the stream function  $\psi$ . The same equation can also be used to replace the time derivatives of density and energy in the compressible flow equations. However, some analysis is required to replace the time derivatives appearing in the Navier-Stokes equations when written in the covariant or contravariant components of the velocity vector.

Let  $\tilde{v}$  be the velocity vector so that  $\frac{\partial \tilde{v}}{\partial t}$  is the partial time derivative term in the vector form of the Navier-Stokes equation. If time dependent coordinates are used then the form of the time derivative becomes

$$(\frac{\partial \tilde{v}}{\partial t})_{\xi^j} + \frac{\partial \tilde{v}}{\partial \xi^k} \frac{\partial \xi^k}{\partial t} , \quad (55)$$

and not simply as  $(\frac{\partial \tilde{v}}{\partial t})_{\xi_j}$ . Here for ease of writing the expressions we have denoted the curvilinear coordinates by  $\xi^j$  ( $j=1, 2$ ) .

Now

$$-\frac{\partial \tilde{v}}{\partial \xi^k} = v_i^k a_i , \quad (56)$$

where  $v^i_k$  is the covariant derivative of the contravariant component  $v^i$ , and  $\underline{a}_i$  represent the covariant base vectors. From (A1-16) and (A1-18), we have

$$v^i_k = \frac{\partial v^i}{\partial \xi^k} + r^i_{kr} v^r . \quad (57)$$

The vector  $\underline{v}$  written in the contravariant components  $v^i$  is

$$\underline{v} = \underline{a}_i v^i \quad (58)$$

Substituting (58) in (55) we get on using (56)

$$\frac{\partial v^i}{\partial t} \underline{a}_i + \frac{\partial \underline{a}_i}{\partial t} v^i + v^i_k \underline{a}_i \frac{\partial \xi^k}{\partial t} . \quad (59)$$

Multiplying each term of (59) scalarly by the contravariant base vector  $\underline{a}^\ell$  and recalling the property

$$\underline{a}_m \cdot \underline{a}^\ell = \delta_m^\ell ,$$

(59) becomes

$$\frac{\partial v^i}{\partial t} + v^k \frac{\partial \underline{a}_k}{\partial t} \cdot \underline{a}^i + v^i_k \frac{\partial \xi^k}{\partial t} , \quad (60)$$

where repeated indices always imply summation.

Denoting  $\xi^1 = \xi$  and  $\xi^2 = \eta$ , the values of  $\frac{\partial \xi^k}{\partial t}$  in (60) are those as given in Eqs. (53). It must be noted that if  $v^i$  is replaced by a scalar function then the middle term in (60) does not appear and the expression coincides exactly with (54).

Based on the equations

$$\underline{a}^i = g^{ij} \underline{a}_j ,$$

and

$$\underline{a}_i = \frac{\partial \underline{r}}{\partial \xi^i} ,$$

we have

$$v^k \frac{\partial \underline{a}_k}{\partial t} \cdot \underline{a}^i = v^1 \frac{\partial \underline{a}_1}{\partial t} \cdot \underline{a}^i + v^2 \frac{\partial \underline{a}_2}{\partial t} \cdot \underline{a}^i .$$

Thus, for  $i = 1$ :

$$v^k \frac{\partial \underline{a}_k}{\partial t} \cdot \underline{a}^1 = v^1 (x_{\xi t} y_\eta - y_{\xi t} x_\eta) / J + v^2 (x_{\eta t} y_\eta - y_{\eta t} x_\eta) / J .$$

For i = 2:

$$v^k \frac{\partial a_k}{\partial t} \cdot a^2 = v^1 (y_{\xi t} x_{\xi t} - x_{\xi t} y_{\xi t}) / J + v^2 (y_{nt} x_{\xi t} - x_{nt} y_{\xi t}) / J.$$

## 6. Generalizations and Extensions

For the sake of clarity of exposition the preceding development has been confined only to doubly connected regions, i.e., the regions formed by a body and an external boundary. The same concepts of coordinate generation are applicable to multiply connected regions. Given any number of bodies, we can join the bodies through simple cuts and the last body can be joined with the external boundary by a cut. The transformation from the physical  $(x, y)$  space to the transformed  $(\xi, \eta)$  space then yields discrete segments on the  $\eta = \text{const.}$  line; each segment representing one body. The computer algorithm for such cases has been thoroughly developed by one of us (JFT) and a detailed discussion is available in Ref. [11].

The most difficult and also rewarding area of extension of the concepts of coordinate generation lies in the three-dimensional situations. It is hoped that the ideas presented in this paper may be of help in clarifying some concepts in three dimensions.

#### REFERENCES

1. Winslow, A. J., "Numerical Solution of the Quasi-Linear Poisson Equation in a Non-Uniform Triangular Mesh," Journal of Computational Physics, 2, 149 (1966).
2. Barfield, W. D., "An Optimal Mesh Generator for Lagrangian Hydrodynamic Calculations in Two Space Dimensions," Journal of Computational Physics, 6, 417 (1970).
3. Chu, W. H., "Development of a General Finite Difference Approximation for a General Domain, Part I: Machine Transformation," Journal of Computational Physics, 8, 392 (1971).
4. Amsden, A. A. and Hirt, C. W., "A Simple Scheme for Generating General Curvilinear Grids," Journal of Computational Physics, 11, 348 (1973).
5. Godunov, S. K. and Prokopov, G. P., "The Use of Moving Meshes in Gas Dynamics Computation," USSR Computational Mathematics and Mathematical Physics, 12, 182 (1972).
6. Thompson, J. F., Thames, F. C., and Mastin, C. W., "Automatic Numerical Generation of Body-Fitted Curvilinear Coordinate System for Field Containing Any number of Arbitrary Two-Dimensional Bodies," Journal of Computational Physics, 15, 299 (1974).
7. Thames, Frank C., "Numerical Solution of the Incompressible Navier-Stokes Equations About Arbitrary Two-Dimensional Bodies," Ph.D. Dissertation, Mississippi State University, (1975).
8. Thames, F. C., Thompson, J. F., and Mastin, C. W., "Numerical Solution of the Navier-Stokes Equations for Arbitrary Two-Dimensional Airfoils," Proceedings of NASA Conference on Aerodynamic Analyses Requiring Advanced Computers, Langley Research Center, NASA SP-347, (1975).

9. Thompson, J. F., Thames, F. C., Mastin, C. W., and Shanks, S. P., "Numerical Solutions of the Unsteady Navier-Stokes Equations for Arbitrary Bodies Using Boundary-Fitted Curvilinear Coordinates," *Proceedings of Arizona/AFOSR Symposium on Unsteady Aerodynamics*, Univ. of Arizona, Tucson, Arizona, (1975).
10. Thompson, J. F., Thames, F. C., and Shanks, S. P., "Use of Numerically Generated Body-Fitted Coordinate Systems for Solution of the Navier-Stokes Equations," *Proceedings of AIAA 2nd Computational Fluid Dynamics Conference*, Hartford, Connecticut, (1975).
11. Thompson, J. F., Thames, F. C., and Mastin, C. W., "Boundary-Fitted Curvilinear Coordinate Systems for Solution of Partial Differential Equations on Fields Containing Any Number of Arbitrary Two-Dimensional Bodies," *NASA Contractor's Report* (1976). To be published.
12. Sokolnikoff, I. S., "Tensor Analysis, Theory and Application," John Wiley and Sons, Inc., (1951).
13. Warsi, Z. U. A., "A Course of Lectures on the Theoretical Aspects of Laminar Viscous Incompressible Flows," published by the Department of Aerophysics and Aerospace Engineering, Mississippi State University, October (1975).
14. Protter, M. H., and Weinberger, H. F., "Maximum Principles in Differential Equations," Prentice-Hall, Inc., (1967).

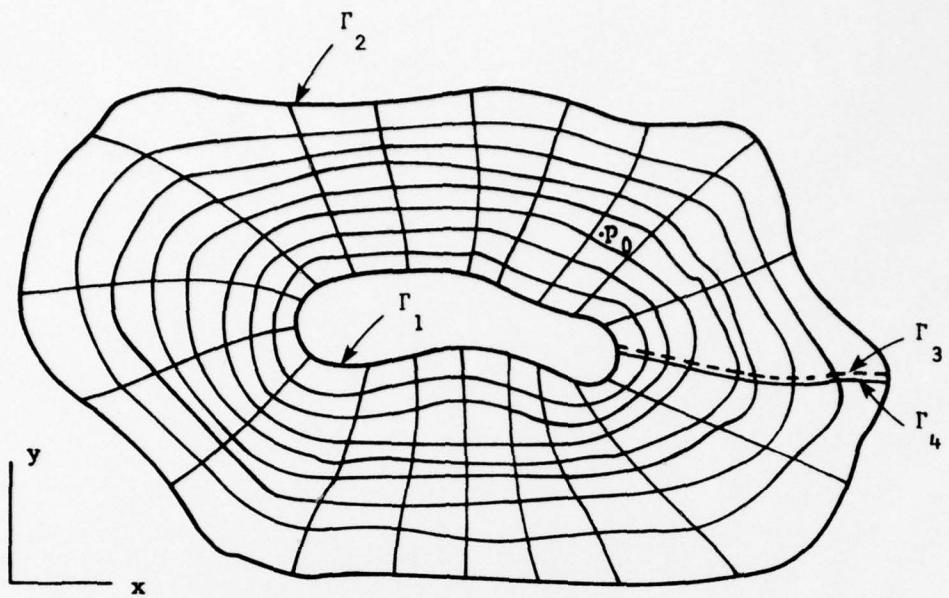
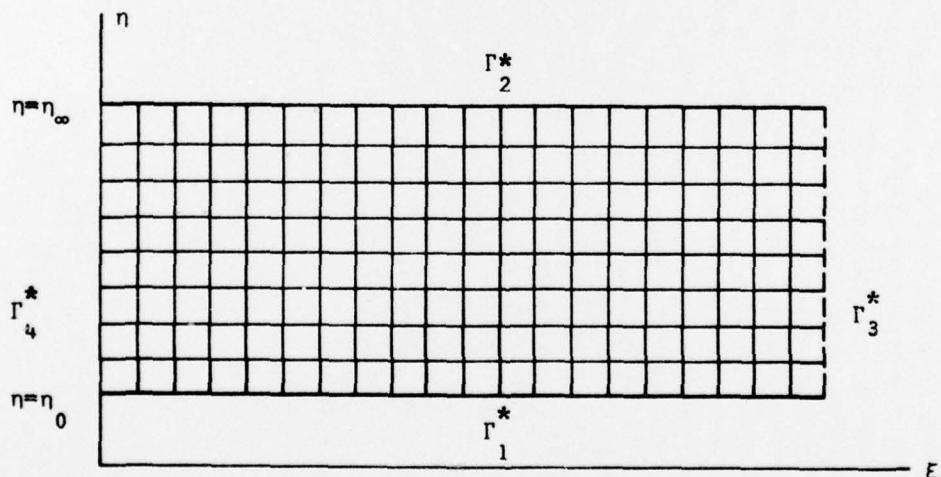
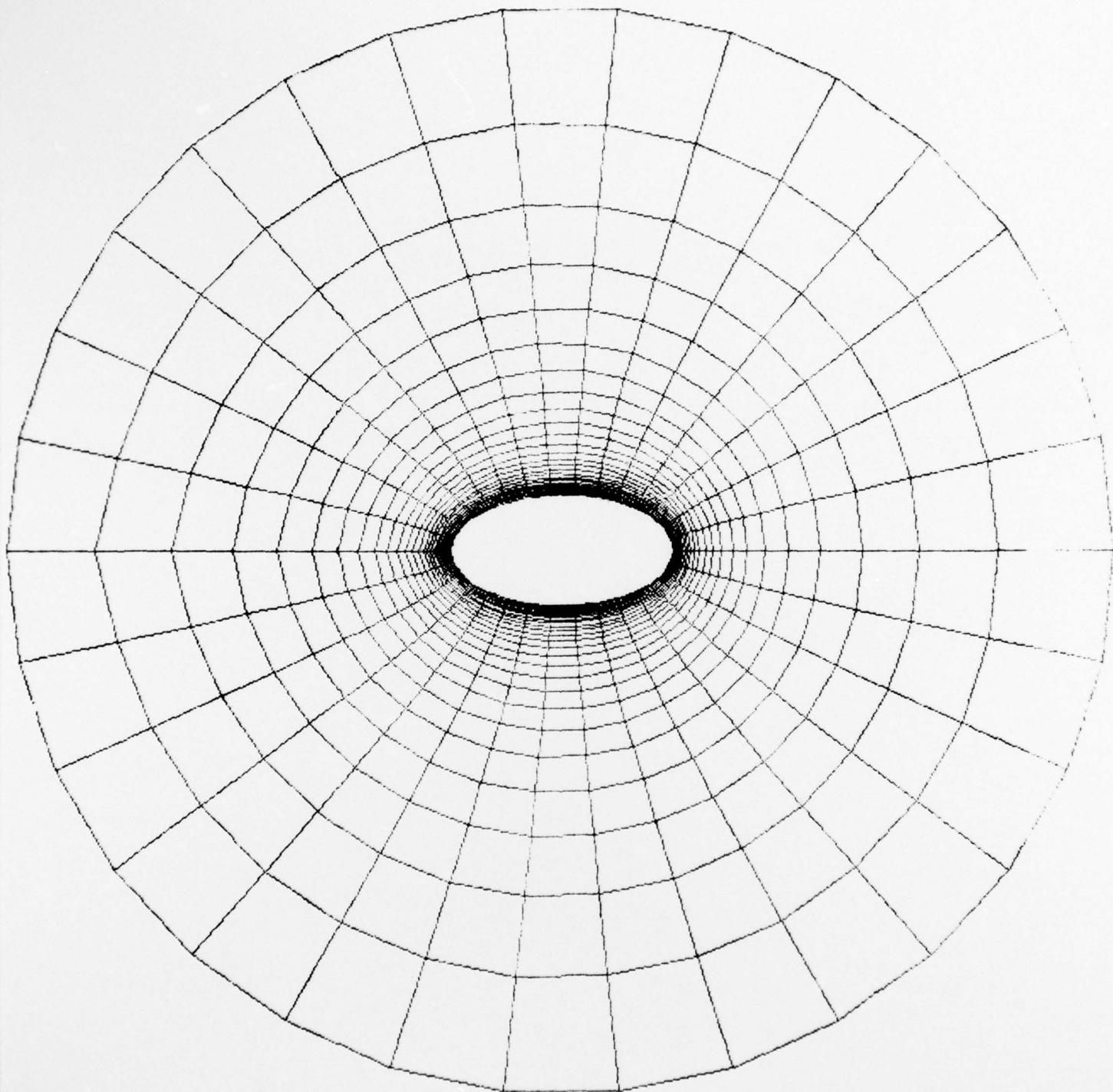


Figure 1. Physical Plane



Transformed Plane  
(Natural Coordinates)

Figure 2. Field Transformation



**Fig. 2** Coordinate Contraction on an Ellipse

## Appendix - A1

In this Appendix we summarize some results from the theory of tensors which are always needed in the formulation of problems in general curvilinear coordinates. For further details refer to Sokolnikoff [12]. Some derivations can also be found in Ref. [13].

### Notation and Definitions

Following the standard practice in tensor theory we shall use variables both in subscript and superscript index notation. In what follows, the repeated indices always imply summation.

In the three-dimensional Euclidean space we introduce a Cartesian coordinate system denoted as  $x_i$ , where  $i=1,2,3$ . Embedded in this Cartesian space, we introduce a general curvilinear coordinate system denoted as  $\xi^i$ , where  $i=1,2,3$ . Thus  $x_i$  are functions of  $\xi^i$  and vice versa.

The magnitude of the displacement vector  $d\xi$  denoted as  $ds$  when referred to the Cartesian coordinates is given by

$$(ds)^2 = \delta_j^i dx_i dx_j , \quad (A1-1)$$

while referred to the general curvilinear coordinates it is given by

$$(ds)^2 = g_{ij} d\xi^i d\xi^j . \quad (A1-2)$$

In (A1-1)  $\delta_j^i$  are the Kronecker deltas defined as

$$\begin{aligned} \delta_j^i &= 1 \quad \text{if } i=j , \\ &= 0 \quad \text{if } i \neq j . \end{aligned} \quad (A1-3)$$

In (A1-2)  $g_{ij}$  are the covariant components of the metric tensor. It is a symmetric tensor, viz.,

$$g_{ij} = g_{ji} . \quad (A1-4)$$

Equating (A1-1) and (A1-2), we can easily establish the formula

$$g_{ij} = \frac{\partial x_k}{\partial \xi^i} \frac{\partial x_k}{\partial \xi^j} \quad (A1-5)$$

The contravariant components of the metric tensor are  $g^{ij}$  where

$$g^{ik} g_{jk} = \delta_j^i . \quad (A1-6)$$

Thus

$$g^{ij} = \frac{\partial \xi^i}{\partial x_k} \frac{\partial \xi^j}{\partial x_k} . \quad (A1-7)$$

The determinant of the matrix  $\{g_{ij}\}$  is denoted as  $g$ ,

$$g = \det(g_{ij}) . \quad (A1-8)$$

The Jacobian determinant of the transformation is

$$J = \sqrt{g} = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi^1, \xi^2, \xi^3)} , \quad (A1-9)$$

where  $J \neq 0$  means that the transformation is not singular.

In the process of differentiation of vectors and tensors in general coordinate systems certain quantities known as the Christoffel symbols appear, which for any values of  $i, j$  and  $k$  are given by

$$\Gamma_{jk}^i = \frac{1}{2} g^{il} \left( \frac{\partial g_{lj}}{\partial \xi^k} + \frac{\partial g_{lk}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^l} \right) . \quad (A1-10)$$

In particular

$$\Gamma_{kr}^k = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^r} (\sqrt{g}) = \frac{1}{2g} \frac{\partial g}{\partial \xi^r} \quad (A1-11)$$

The quantities  $\Gamma_{jk}^i$  are symmetric in the lower two indices,

$$\Gamma_{jk}^i = \Gamma_{kj}^i . \quad (A1-12)$$

#### Differentiation of Vectors and Tensors

In general coordinates the subscripts denote the covariant components while the superscripts denote the contravariant components. In Cartesian coordinates since there is no distinction between the covariant and contravariant components, the indices have no special meaning. Thus the subscript on  $x$  in all the previous formulae is merely a symbol and should not be considered a covariant index. Below we give some examples.

Let  $a$  be a vector, then it can either be expressed in terms of its covariant components  $a_i$  or the contravariant components  $a^i$ .

Let  $A$  be a tensor of second order. It can be expressed in three

distinct ways:

- $A_{ij}$ , the covariant components,
- $A^{ij}$ , the contravariant components,
- $A_j^i$ , the mixed covariant-contravariant components.

The displacement vector  $\underline{dr}$  has components  $dx_i$ , or  $(dx_1, dx_2, dx_3)$  in the Cartesian coordinates without any significance attached to the subscripts. On the other hand  $\underline{dr}$  has the contravariant components  $d\xi^i$ , or  $(d\xi^1, d\xi^2, d\xi^3)$  in any general coordinate system and actually this was the reason for denoting the general coordinates by a superscript, though  $\xi^i$  themselves are not the contravariant components. Thus

$$\underline{dr} = \frac{\partial \underline{r}}{\partial \xi^i} d\xi^i = a_i d\xi^i . \quad (A1-13)$$

The quantities  $a_i$ ,  $i=1,2,3$ , are called the covariant base vectors. Obviously

$$g_{ij} = a_i \cdot a_j . \quad (A1-14)$$

Let  $\phi$  be a scalar function, then

$$\underline{b} = \text{grad } \phi$$

is a vector whose covariant components in any coordinate system are

$$b_i = \frac{\partial \phi}{\partial \xi^i} \quad (A1-15)$$

Let  $\underline{v}$  be a vector, then its gradient is a second order tensor denoted as

$$\underline{\underline{A}} = \text{grad } \underline{v} .$$

The mixed components of  $\underline{\underline{A}}$  are given by

$$A_k^i = \frac{\partial v^i}{\partial \xi^k} + \Gamma_{kr}^i v_r . \quad (A1-16)$$

The covariant components of  $\underline{\underline{A}}$  are given by

$$A_{ik} = \frac{\partial v_i}{\partial \xi^k} - \Gamma_{ik}^r v_r . \quad (A1-17)$$

The right hand sides of (A1-16) and (A1-17) are called the covariant derivatives of the contravariant and covariant vector components respectively.

The covariant derivative is always denoted by a coma (,) preceding an

index. Thus in (Al-16), (Al-17)

$$A_k^i = v_{,k}^i , \quad (Al-18)$$

$$A_{ik,l} = v_{i,k,l}^i \quad (Al-19)$$

Covariant differentiation is also called absolute differentiation because it holds in any coordinate system including the Cartesian system where the Christoffel symbols are identically zero. The divergence of a vector is obtained by setting  $k=i$  in (Al-18) and then invoking the summation convention.

Thus

$$\text{div } \underline{v} = v_{,i}^i \quad (Al-20)$$

The formulae for the covariant differentiation of second order tensors are

$$A_{ik,\ell} = \frac{\partial A_{ik}}{\partial \xi^\ell} - \Gamma_{il}^r A_{rl} - \Gamma_{kl}^r A_{ir} , \quad (Al-21)$$

$$A_{k,\ell}^i = \frac{\partial A_k^i}{\partial \xi^\ell} + \Gamma_{lr}^i A_{k,r}^r - \Gamma_{kl}^r A_r^i , \quad (Al-22)$$

$$A_{,\ell}^{ik} = \frac{\partial A^{ik}}{\partial \xi^\ell} + \Gamma_{\ell r}^i A^{rk} + \Gamma_{\ell r}^k A^{ir} . \quad (Al-23)$$

A theorem due to Ricci states (which can be proved quite simply by using (Al-21) - (Al-23)) that the covariant derivatives of the metric tensor are zero. That is

$$g_{ij,k} = 0 , \quad g_{,k}^{ij} = 0 . \quad (Al-24)$$

Based on the preceding formulae, we state the following results.

Let  $\underline{v}$  be an arbitrary vector, then

$$\text{div } \underline{v} = v_{,i}^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} (\sqrt{g} v^i) . \quad (Al-25)$$

Let  $T$  be an arbitrary second order tensor. Then its divergence is a vector. Writing

$$\underline{b} = \text{div } \tilde{T} ,$$

the covariant components of  $\underline{b}$  are given by

$$b_i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^k} (\sqrt{g} T_i^k) - \Gamma_{ir}^s T_s^r . \quad (Al-26)$$

The contravariant components are given by

$$b^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^k} (\sqrt{g} T^{ik}) + \Gamma_{rs}^i T^{rs} . \quad (A1-27)$$

Let  $\phi$  be a scalar, then its Laplacian in general coordinates is given by

$$\nabla^2 \phi = g^{jk} \left( \frac{\partial^2 \phi}{\partial \xi^j \partial \xi^k} - \Gamma_{jk}^r \frac{\partial \phi}{\partial \xi^r} \right) . \quad (A1-28)$$

For an arbitrary vector, the following formulae establish the relations between the covariant and contravariant components

$$v_i = g_{ij} v^j , \quad v^i = g^{ij} v_j \quad (A1-29)$$

#### Expressions In Two Dimensions

In two dimensions both  $i$  and  $j$  vary from 1 to 2. We shall write

$$\xi^1 = \xi, \quad \xi^2 = \eta, \quad x_1 = x, \quad x_2 = y . \quad (A1-30)$$

Denoting the partial derivatives by variable subscripts we have

$$g_{11} = x_\xi^2 + y_\xi^2 ,$$

$$g_{12} = x_\xi x_\eta + y_\xi y_\eta ,$$

$$g_{22} = x_\eta^2 + y_\eta^2 ,$$

$$g = g_{11} g_{22} - (g_{12})^2 = (x_\xi y_\eta - x_\eta y_\xi)^2 ,$$

$$J = \sqrt{g} ,$$

$$g^{11} = g_{22}/g , \quad g^{12} = g^{21} = -g_{12}/g , \quad g^{22} = g_{11}/g . \quad (A1-31)$$

The Christoffel symbols are

$$\Gamma_{11}^1 = [g_{22} \frac{\partial g_{11}}{\partial \xi} + g_{12} (\frac{\partial g_{11}}{\partial \eta} - 2 \frac{\partial g_{12}}{\partial \xi})]/2g ,$$

$$\Gamma_{22}^2 = [g_{11} \frac{\partial g_{22}}{\partial \eta} + g_{12} (\frac{\partial g_{22}}{\partial \xi} - 2 \frac{\partial g_{12}}{\partial \eta})]/2g ,$$

$$\Gamma_{22}^1 = [g_{22} (2 \frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) - g_{12} \frac{\partial g_{22}}{\partial \eta}]/2g ,$$

$$\Gamma_{11}^2 = [g_{11} (2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}) - g_{12} \frac{\partial g_{11}}{\partial \xi}]/2g ,$$

$$\Gamma_{12}^1 = \Gamma_{21}^1 = (g_{22} \frac{\partial g_{11}}{\partial \eta} - g_{12} \frac{\partial g_{22}}{\partial \xi})/2g ,$$

$$\begin{aligned}\Gamma_{12}^2 &= \Gamma_{21}^2 = (g_{11} \frac{\partial g_{22}}{\partial \xi} - g_{12} \frac{\partial g}{\partial \eta})/2g \quad , \\ \Gamma_{11}^1 + \Gamma_{12}^2 &= \frac{1}{2g} \frac{\partial g}{\partial \xi} \quad , \\ \Gamma_{12}^1 + \Gamma_{22}^2 &= \frac{1}{2g} \frac{\partial g}{\partial \eta} \quad .\end{aligned}\tag{A1-32}$$

The Laplacian of  $\phi$  is

$$\begin{aligned}\nabla^2 \phi &= [g_{22} \phi_{\xi\xi} - 2g_{12} \phi_{\xi\eta} + g_{11} \phi_{\eta\eta} \\ &\quad + (2g_{12} \Gamma_{12}^1 - g_{22} \Gamma_{11}^1 - g_{11} \Gamma_{22}^1) \phi_\xi \\ &\quad + (2g_{12} \Gamma_{12}^2 - g_{22} \Gamma_{11}^2 - g_{11} \Gamma_{22}^2) \phi_\eta]/g \quad .\end{aligned}\tag{A1-33}$$

## Appendix - A2

### Criteria For Elliptic Equations

Let us consider two coupled nonlinear partial differential equations written generally with dependent variables  $x$  and  $y$  as

$$F_1(\xi, \eta, x, y, x_\xi, x_\eta, y_\xi, y_\eta, x_{\xi\xi}, x_{\xi\eta}, x_{\eta\eta}) = 0 , \quad (A2-1)$$

$$F_2(\xi, \eta, x, y, x_\xi, x_\eta, y_\xi, y_\eta, y_{\xi\xi}, y_{\xi\eta}, y_{\eta\eta}) = 0 , \quad (A2-2)$$

where  $x = x(\xi, \eta)$  and  $y = y(\xi, \eta)$ . For brevity we write

$$x_\xi = p_1, \quad y_\xi = p_2, \quad x_\eta = q_1, \quad y_\eta = q_2 ,$$

$$x_{\xi\xi} = r_1, \quad y_{\xi\xi} = r_2, \quad x_{\xi\eta} = s_1, \quad y_{\xi\eta} = s_2 ,$$

$$x_{\eta\eta} = t_1, \quad y_{\eta\eta} = t_2 .$$

If for all pairs of real numbers  $(\mu, v)$  such that  $\mu^2 + v^2 > 0$  we have

$$\frac{\partial F_1}{\partial r_1} \mu^2 + \frac{\partial F_1}{\partial s_1} \mu v + \frac{\partial F_1}{\partial t_1} v^2 > 0 , \quad (A2-3)$$

and

$$\frac{\partial F_2}{\partial r_2} \mu^2 + \frac{\partial F_2}{\partial s_2} \mu v + \frac{\partial F_2}{\partial t_2} v^2 > 0 , \quad (A2-4)$$

then the partial differential equations (A2-1) and (A2-2) are called elliptic.

The inequalities (A2-3) and (A2-4) hold both for linear and nonlinear differential equations.

As an application we consider Eqs. (13) and (14) and apply (A2-3) and (A2-4). This gives

$$g_{22}\mu^2 - 2g_{12}\mu v + g_{11}v^2 > 0 . \quad (A2-5)$$

Using the definitions of the metrical coefficients (A1-31), we get

$$(\mu q_1 - v p_1)^2 + (\mu q_2 - v p_2)^2 > 0 , \quad (A2-6)$$

which shows that Eqs (13) and (14) are uniformly elliptic throughout out the domain.

### Appendix - A3

#### Harmonic, Subharmonic and Superharmonic Functions

Let  $u(x,y)$  be a continuous function in the closed domain  $\bar{D}=D \cup \partial D$  and twice continuously differentiable in the open domain  $D$ . Consider the equation

$$\nabla^2 u = F \quad \text{in } D, \quad (\text{A3-1})$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (\text{A3-2})$$

If  $F = 0$ , then the function  $u$  is called a harmonic function while if  $F \geq 0$ , then  $u$  is called a subharmonic function.

#### Maximum Principle:

Consider Eq. (A3-1) with  $F \geq 0$ . From elementary calculus we know that if a function  $\omega(x,y)$  attains a maximum at  $(x_0, y_0)$  then

$$\omega_x = 0, \omega_y = 0,$$

$$\omega_{xx} \leq 0, \omega_{yy} \leq 0 \quad \text{at } (x_0, y_0),$$

here variable subscripts denote partial differentiation. Thus, for a maxima,

$$\nabla^2 \omega \leq 0. \quad (\text{A3-3})$$

Let  $\phi(x,y)$  be a non-negative function defined as

$$\phi = \frac{1}{4}\epsilon(x^2+y^2), \quad \epsilon > 0.$$

Defining

$$\omega = u + \phi,$$

we find that

$$\nabla^2 \omega = F + \epsilon \geq 0,$$

since both  $F$  and  $\epsilon$  are non-negative. This contradicts the assertion of a maxima, i.e. (A3-3). Consequently  $\omega$  can attain its maxima only at the boundary  $\partial D$ . Also,

$$\omega = u + \phi \leq M + \frac{1}{4}\epsilon\rho^2,$$

where  $\rho$  is the radius of a circle containing  $D$ . Since  $u \leq \omega$ , hence

$$u \leq M + \frac{1}{4}\epsilon\rho^2.$$

Letting  $\epsilon \rightarrow 0$ , we find that  $u \leq M$  in  $D$ .

In conclusion we state that if  $u$  satisfies (A3-1) and  $F \geq 0$ , then the values of  $u$  in  $D$  cannot exceed the maximum on  $\partial D$ .

For harmonic functions, viz., when  $F=0$ , we can apply the above deductions both to  $u$  and  $-u$ . This leads to state that a harmonic function can attain either its maximum or its minimum only at the boundary  $\partial D$ .

#### Subharmonic Functions:

The values of a subharmonic function  $u$  in a domain  $D$  are always below the values of the harmonic function  $v$  which coincides with  $u$  on  $\partial D$ . For a proof of this assertion we follow Protter and Weingerer [14].

Define

$$\omega = u - v ,$$

where  $u$  is subharmonic,  $v$  is harmonic, and  $v=u$  on  $\partial D$ . Thus  $\omega$  is subharmonic and vanishes on  $\partial D$ . According to the maximum principle  $\omega \leq 0$  in  $D$ . Thus

$$u \leq v \text{ in } D,$$

which proves the assertion.

#### Superharmonic Functions:

For the case when  $F \leq 0$  we similarly establish a minimum principle which states that a superharmonic function  $u$  cannot attain a minimum below the values of a harmonic function  $v$  which coincides with  $u$  on  $\partial D$ .

SECURITY CLASSIFICATION OF THIS PAGE (If different from Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER <i>AFOSR IR-76-1211</i>	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) MACHINE SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS IN THE NUMERICALLY GENERATED COORDINATE SYSTEMS	5. TYPE OF REPORT & PERIOD COVERED Interim	
7. AUTHOR(s) Z. U. A. Warsi J. F. Thompson Jr	8. CONTRACT OR GRANT NUMBER(s) AFOSR 76-2922 <i>new</i>	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mississippi State University Aerophysics & Aerospace Engineering Mississippi State, Mississippi 39762	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A3	
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research (NM) Bldg 410, Bolling AFB, DC 20332	12. REPORT DATE August 1976	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	13. NUMBER OF PAGES 32	
15. SECURITY CLASS. (of this report) UNCLASSIFIED		
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Numerical Solution of Partial Differential Equations Numerical Generation of General Coordinates Numerical Solution of Field Equations Navier-Stokes Solutions		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper presents a study in depth on the numerically generated body-fitted coordinate systems. All fundamental ideas have been developed from a very basic point of view which leads to a well structured method of coordinate system control in any region of interest. It is also shown how the developed concepts are instrumental in obtaining the machine solutions of the field differential equations in general dependent variables on the generated coordinate systems.		

